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# A lower bound for ground-state energy by Steiner symmetrisation of the potential

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**Abstract.** It is shown that Steiner symmetrisation of the potential leads to a lowering of the ground-state energy of the Hamiltonian. Some special cases in which Steiner symmetrisation of the potential leads to the lowering of *all* the bound-state energy levels of the system are presented. A connection between certain exactly solvable potentials via Steiner symmetrisation is also brought out. Counterexamples are given to show that, in general, the lowering of energy eigenvalues on Steiner symmetrisation does not occur for *all* levels.

## 1. Introduction

A process of symmetrisation of geometrical objects in all dimensions, now known as Steiner symmetrisation, was introduced by Jacob Steiner in 1836 in connection with his investigations on the so-called 'great isoperimetric theorem'—of all closed planar curves with a given perimeter, the circle has the largest area. This symmetrisation enabled Steiner to considerably generalise the above theorem in various directions and to higher dimensions. For example, he showed that a planar polygon of a given number of sides and given perimeter has the largest area when it is a regular polygon.

Comparatively recently, in the 1940s, Polya and Szegö applied Steiner's idea to certain isoperimetric problems in physics for the estimation of 'not easily accessible physical quantities on the basis of easily accessible geometrical data'. Application of Steiner symmetrisation enabled them to make substantial contributions towards the proofs of a large number of conjectures dating from that of Saint-Venant on the torsional rigidity of a planar membrane, Rayleigh on the principal frequency of a vibrating membrane of arbitrary shape and on the electrostatic capacity of a planar plate and similar conjectures of Minkowski, Poincaré and others. The fruitfulness of the idea can be gauged by the fact that Polya and Szegö were able to provide a unified treatment of most of these problems on which the earlier investigators could achieve only partial success by using different techniques for different problems.

More recently, Landau and Lifshitz (1960) derived a very interesting result concerning potentials having the same Steiner symmetrisation. They were solving the 'inverse problem' of classical mechanics—of finding the potential from the known form of the period of oscillation  $T$  as a function of its total energy  $E$ . They showed that the solution is unique provided the potential has a single minimum and that it is symmetric with respect to the location of the minimum. Anticipating the definition of Steiner

symmetrisation given later in this paper, what their result means is that all single-minimum potentials having the same Steiner symmetrised form lead to the same  $T(E)$ . Such potentials have therefore been called 'isoperiodic potentials' by Katriel and Rosenhouse (1985). Again, without explicitly using the concept of Steiner symmetrisation, Wheeler (1976) has established that every potential in this class has the same eigenvalue spectrum in the wkb approximation. It should be noted that all these authors implicitly or explicitly consider only single-minimum potentials. Katriel and Rosenhouse (1985) have studied the quantum mechanical spectra of several families of classically isoperiodic potentials. There is considerable literature (see, for example, Nieto 1981, Nieto and Simmons 1979, Nieto and Gutschick 1981 Ghosh and Hasse 1981) on families of potentials having the same quantum mechanical energy spectra (isospectral potentials) or the same classical period of oscillation. The two classes are not identical, though they have a non-null intersection.

In the present paper, we show, following closely the method used by Polya and Szegö (1951), that Steiner symmetrisation of the potential energy function leads to a lowering of the ground-state eigenvalue of the Schrödinger equation. Since symmetrisation generally leads to a simplification in the numerical evaluation of the eigenvalue, this lower bound on the actual eigenvalue, together with a variational upper bound, could be useful in obtaining a reasonably good estimate of the ground-state energy. In this connection we may also mention that a lower bound for the ground-state energy for a wide class of potentials has been established by Barnes *et al* (1976) using a generalised form of Young's inequality obtained independently by Beckner (1975) and Brascamp and Lieb (1976). They have applied their bound to the power potentials  $V(x) = C|x|^\gamma$ , with the constants  $C > 0$ ,  $\gamma > 0$ .

We also give a simple argument to show that the lowering of the eigenvalue occurs, in general, only for the ground state and not for the excited states. In a number of special cases (in one dimension), Steiner symmetrisation leads to a lowering of *all* levels. These examples are interesting because they illustrate how several different exactly solvable potentials are connected with one another through Steiner symmetrisation.

Accordingly, in § 2 we briefly recall the definition of Steiner symmetrisation and recapitulate some pertinent results. Using these we establish the main result of this paper: Steiner symmetrisation of the potential lowers the ground-state energy of a quantum mechanical system. In § 3 we present a few examples where exact solutions are possible for both the Hamiltonians, before and after the Steiner symmetrisation of the potential. Here we find that all the energy levels are lowered on Steiner symmetrisation of the potential. Counterexamples are presented in § 4 to show that the lowering occurs, in general, only for the ground-state energy and not for the excited states. There we consider the Schrödinger equation with two  $\delta$  function potentials. Under suitable conditions the corresponding Hamiltonian has two bound states. Steiner symmetrisation leads to a single  $\delta$  function potential of appropriate strength having only one bound state. Thus the Steiner symmetrised potential has only one bound state, whereas the original potential can have two bound states. Hence in this case the first excited level of the potential after Steiner symmetrisation (corresponding to the lowest level of the positive energy continuum) is higher than the (only) excited bound-state energy level in the potential before symmetrisation. Another counterexample is provided by comparing the exact eigenenergies of an asymmetric oscillator obtained by Ghosh and Hasse (1981) with those of its Steiner symmetrised form. Conclusions are summarised in § 5.

## 2. Steiner symmetrisation of a function

Steiner's original idea of symmetrisation pertained to geometrical objects. Hence the suggested construction for symmetrisation was based on geometry. For our present purpose, however, we will need an analytical definition of Steiner symmetrisation. With the two kinds of symmetrisations defined below (following Polya and Szegö 1951) (see also Hardy *et al* 1951) one cannot only recover Steiner's original idea of symmetrisation of geometrical objects, but also give a proper characterisation of symmetrisation of open surfaces (open curves) required in the present work.

We shall first consider Steiner symmetrisation of functions of a single real variable. Let  $f(x)$  be a piecewise continuous function of a real variable  $x$  defined over the interval  $(a, b)$  and vanishing at its endpoints. Let  $M(y)$  be the measure of the set of points  $\{x|f(x) \geq y\}$ . Then the Steiner symmetrised function  $f^+(x)$  corresponding to  $f(x)$  is defined as (Polya and Szegö 1951)

$$f^+(\frac{1}{2}M(y)) = y \quad f^+(-x) = f^+(x). \tag{2.1}$$

$f^+(x)$  is a symmetrically decreasing function of  $x$ . In a similar manner, we define a symmetrically increasing function  $f^-(x)$ :

$$f^-(\frac{1}{2}m(y)) = y \quad f^-(-x) = f^-(x) \tag{2.2}$$

where  $m(y)$  is the measure of the set of points  $\{x|f(x) \leq y\}$ . It is easy to see that the two symmetrised functions defined above are related as

$$f^-(x) = -[-f(x)]^+. \tag{2.3}$$

Generalisation to functions of many variables is straightforward. Thus, if  $f(x, y, z)$  is a function of three variables  $x, y$  and  $z$ , Steiner symmetrisation is obtained, for example, by considering  $f$  to be a function of  $z$  for fixed  $x$  and  $y$  (Polya and Szegö 1951). This corresponds to symmetrisation of the function with respect to the  $xy$  plane. The function may be symmetrised with respect to other planes as well. We shall now recall some useful results pertaining to Steiner symmetrisation.

(i) If  $f \geq 0$ , it is easy to see from the definitions that

$$[f^2]^+ = [f^+]^2. \tag{2.4}$$

(ii) For  $f(x) \geq 0$ , let  $M^+(y)$  be the measure of the set of points  $\{x|f^+(x) \geq y\}$ . Then from equation (2.1) we have  $M^+(y) = M(y)$ . It follows that

$$\int f(x) dx = \int M(y) dy = \int M^+(y) dy = \int f^+(x) dx \tag{2.5}$$

where the integrations are performed over the respective domains. The generalisation to functions of more than one variable is obvious.

(iii) For a non-negative function  $f$  vanishing outside a certain domain we have

$$\int |\nabla f|^2 dx dy dz \geq \int |\nabla f^+|^2 dx dy dz. \tag{2.6}$$

The proof of the inequality (see Polya and Szegö 1951) makes use of the fact that under Steiner symmetrisation the volume of a solid remains constant but its surface area diminishes (does not increase). An analogous inequality for the case of spherical (or 'Schwarz') symmetrisation has been derived independently by Glaser *et al* (1976). Their inequality, at least intuitively, is a consequence of the above inequality (for Steiner symmetrisation), since spherical symmetrisation is achieved by an infinite

sequence of Steiner symmetrisations with respect to suitably chosen 'planes' of symmetrisation.

(iv) For functions  $f$  and  $g$  the following inequality holds (Hardy *et al* 1951):

$$\int fg \, dx \, dy \, dz \geq \int f^+ g^- \, dx \, dy \, dz. \quad (2.7)$$

The above inequality is the extension of the following result which holds for finite-dimensional vector spaces. Suppose that only the components (but not their order) of two vectors  $a$  and  $b$  are given. Then their scalar product is largest if the components of  $a$  and  $b$  are arranged in the same order (for example, both increasing or both decreasing) and least if they are arranged in the opposite order. (Note that the function  $f^+$  is symmetrically decreasing and  $g^-$  is symmetrically increasing, i.e. they are oppositely ordered.)

The above inequalities remain valid even when the domains of definition of the functions are infinite, provided the relevant integrals exist.

We are now in a position to prove the main result of this paper: if  $E_0$  is the ground-state energy of the Hamiltonian  $H = p^2/2m + V$  and  $E_0^-$  is the ground-state energy corresponding to the Hamiltonian  $H^- = p^2/2m + V^-$  obtained by Steiner symmetrisation of the potential  $V$ , then  $E_0 \geq E_0^-$ .

It may be noted that, for a confining potential  $V(x)$ ,  $-\infty < x < \infty$ ,  $V^-(x)$  will also be confining but  $V^+(x)$  will not be so. This can easily be checked for the potential in the form of a well with a single minimum and bounded above. The Steiner symmetrisation  $V^-(x)$  will also be a well similar to  $V(x)$  but  $V^+(x)$  will be in the form of a barrier (with a single maximum) and will not sustain any bound state.

The above theorem is the quantum mechanical analogue of the corresponding result for the fundamental frequency of a vibrating membrane and other similar classical results. We give below a simple proof of the theorem as it does not seem to have been explicitly given elsewhere in the literature.

The proof follows from the inequalities stated above and the variational principle. Let  $\psi$  be the normalised exact ground-state eigenfunction of the Hamiltonian  $H$ . It will have a maximum and will vanish at  $x = \pm\infty$ . The function  $\psi^-$  does not satisfy the required boundary conditions, in order that it represent a bound state of a physical system. On the other hand, the function  $\psi^+$  has a behaviour similar to  $\psi$ . Furthermore, using the property (2.5) we see that  $\psi^+$  is also normalised. Thus we have

$$\begin{aligned} E_0 &= \int \psi H \psi \, d\tau \\ &= -(\hbar^2/2m) \int \psi \nabla^2 \psi \, d\tau + \int V \psi^2 \, d\tau \\ &\geq (\hbar^2/2m) \int |\nabla \psi|^2 \, d\tau + \int V^-(\psi^2)^+ \, d\tau \quad (\text{using (2.7)}) \\ &\geq (\hbar^2/2m) \int |\nabla \psi^+|^2 \, d\tau + \int V^-(\psi^+)^2 \, d\tau \quad (\text{using (2.6)}) \\ &= -(\hbar^2/2m) \int \psi^+ \nabla^2 \psi^+ \, d\tau + \int V^- \psi^{+2} \, d\tau \\ &= \int \psi^+ H^- \psi^+ \, d\tau \end{aligned}$$

$$\geq E_0^- \quad (\text{by the variational principle}).$$

Thus we see that the ground-state energy of a Hamiltonian with the potential  $V(x)$  is lowered if we replace the potential by the Steiner symmetrised function  $V^-(x)$ .

### 3. Exactly solvable examples

In this section we shall present some examples of potentials  $V(x)$ , capable of binding a particle, for which an exact solution of the Schrödinger equation for both  $V(x)$  as well as  $V^-(x)$  is possible. In all these examples we find that every eigenvalue of the Hamiltonian with potential  $V(x)$  is higher than the corresponding eigenvalue with potential  $V^-(x)$ . All the potentials we consider have been treated elsewhere for different purposes either as examples of 'quantum harmonic' potentials, which lead to equispaced energy eigenvalues, or of 'classical harmonic' potentials, which lead to energy independent periods of oscillation.

#### 3.1. The asymmetric harmonic oscillator

The exact eigenenergies of the Schrödinger equation for the asymmetric harmonic oscillator were recently obtained numerically by Ghosh and Hasse (1981). They demonstrated that, although an asymmetric harmonic oscillator is classical harmonic, it is not quantum harmonic. The potential for this problem may be represented as

$$\begin{aligned} V(x) &= \frac{1}{2}m\omega_1^2x^2 & x > 0 \\ V(x) &= \frac{1}{2}m\omega_2^2x^2 & x < 0. \end{aligned} \tag{3.1}$$

To obtain  $V^-(x)$  corresponding to the above potential we observe that for a given  $y \geq 0$  the measure  $m(y)$  of the set of points  $\{x | V(x) \leq y\}$  is  $x_1 - x_2$  ( $x_1 > 0$  and  $x_2 < 0$ ), where

$$y = \frac{1}{2}m\omega_1^2x_1^2 = \frac{1}{2}m\omega_2^2x_2^2. \tag{3.2}$$

Hence we obtain

$$x_1 = (2y/m)^{1/2}\omega_1^{-1} > 0 \quad x_2 = -(2y/m)^{1/2}\omega_2^{-1} < 0. \tag{3.3}$$

By definition we have

$$V^-((x_1 - x_2)/2) = y \tag{3.4}$$

which gives

$$V^-(2y/m\omega^*{}^2)^{1/2} = y \tag{3.5}$$

where  $\omega^* = 2\omega_1\omega_2/(\omega_1 + \omega_2)$ . Setting  $(2y/m\omega^*{}^2)^{1/2} = x$  we obtain

$$y = \frac{1}{2}m\omega^*{}^2x^2. \tag{3.6}$$

Therefore we obtain, from (3.5) and (3.6),

$$V^-(x) = \frac{1}{2}m\omega^*{}^2x^2 \quad -\infty < x < \infty \tag{3.7}$$

corresponding to a (symmetric) harmonic oscillator with frequency  $\omega^*$ , leading to an energy independent period of oscillations in agreement with the earlier result (Ghosh and Hasse 1981). Since the energy levels of an asymmetrical oscillator cannot be

obtained analytically in a closed form, we consider a limiting case where  $\omega_2 \rightarrow \infty$ . (The case with  $\omega_1$  and  $\omega_2$  finite will be considered in § 4.2.) In the limiting case presently under consideration the asymmetric oscillator reduces to an oscillator on a half-line  $x > 0$ , i.e.

$$\begin{aligned} V(x) &= \infty & x < 0 \\ &= \frac{1}{2}m\omega_1^2 x^2 & x > 0. \end{aligned} \quad (3.8)$$

In the limit  $\omega_2 \rightarrow \infty$ ,  $\omega^* \rightarrow 2\omega_1$  and therefore the limiting form of  $V^-(x)$  is given by

$$V^-(x) = \frac{1}{2}(2\omega_1)^2 x^2 \quad -\infty < x < \infty. \quad (3.9)$$

The eigenenergies corresponding to the potential (3.8) are given by

$$E_n = (2n + \frac{3}{2})\hbar\omega_1 \quad n = 0, 1, 2, \dots \quad (3.10)$$

On the other hand, the energy levels corresponding to  $V^-(x)$  of equation (3.9) are given by

$$E_n^- = (2n + 1)\hbar\omega_1 \quad n = 0, 1, 2, \dots \quad (3.11)$$

From equations (3.10) and (3.11) it is evident that  $E_n > E_n^-$  for every  $n$ .

### 3.2. Harmonic oscillator with inverse square potential (isotonic oscillator)

Let the potential corresponding to this problem be given by

$$V(x) = a^2 x^2 + b^2/x^2 \quad x > 0. \quad (3.12)$$

Writing  $ab = V_0$  and  $c = (b/a)^{1/2}$  we can rewrite (3.12) in the form

$$V(x) = V_0 \left( \frac{x}{c} - \frac{c}{x} \right)^2 + 2V_0. \quad (3.13)$$

The eigenenergies for a particle moving in the potential (3.13) are well known (see, for example, Gol'dman and Krivchenkov 1961) and are given by

$$E_n = \hbar\omega \left\{ n + \frac{1}{2} + \frac{1}{4}[(U_0 + 1)^{1/2} - (U_0)^{1/2}] \right\} + 2V_0 \quad (3.14)$$

where

$$\omega = (8V_0/mc^2)^{1/2} \quad U_0 = (8mV_0c^2/\hbar^2)^{1/2}. \quad (3.15)$$

The Steiner symmetrised form  $V^-(x)$  corresponding to  $V(x)$  of equation (3.12) may be obtained by following the procedure outlined in the previous example or more simply by noting that  $V(x)$  is classical harmonic with the classical angular frequency  $\omega$  given by equation (3.15) (Nieto 1981). Thus  $V^-(x)$  is given by

$$V^-(x) = \frac{1}{2}m\omega^2 x^2 + 2V_0. \quad (3.16)$$

The energy levels corresponding to the above potential are given by

$$E_n^- = (n + \frac{1}{2})\hbar\omega + 2V_0. \quad (3.17)$$

A comparison of the above expression with (3.14) shows  $E_n > E_n^-$  for all  $n$ .

### 3.3. The Morse potential

We shall now compare the energy levels corresponding to the Morse potential

$$V(x) = A(e^{-2\alpha x} - 2e^{-\alpha x}) \quad -\infty < x < \infty \quad (3.18)$$

with those corresponding to its Steiner symmetrised form. The Schrödinger equation with the potential  $V(x)$  given above can be solved exactly (Landau and Lifshitz 1958). The energy eigenvalues  $E_n$  are

$$E_n = -\gamma^2[\gamma^{-1}\sqrt{A} - (2n+1)]^2 \quad (3.19)$$

where  $\gamma = \hbar^2\alpha^2/8m$ . To get  $V^-(x)$  corresponding to (3.18) we proceed as in the earlier example and determine  $x_1$  and  $x_2$  ( $x_1 > x_2$ ) for a given  $y$  such that

$$y = V(x_1) = V(x_2). \quad (3.20)$$

Substituting the form (3.18) for  $V(x)$  we obtain, after some simplification,

$$x_1 = -\alpha^{-1} \ln\{1 - [1 + (y/A)]^{1/2}\} \quad (3.21)$$

$$x_2 = -\alpha^{-1} \ln\{1 + [1 + (y/A)]^{1/2}\}. \quad (3.22)$$

Hence we have

$$(x_1 - x_2)/2 = (2\alpha)^{-1} \ln\left(\frac{1 - [1 + (y/A)]^{1/2}}{1 + [1 + (y/A)]^{1/2}}\right). \quad (3.23)$$

By definition  $V^-((x_1 - x_2)/2) = y$ . Setting  $(x_1 - x_2)/2 = x$  in (3.23) and solving for  $y$  in terms of  $x$  we obtain

$$y \equiv V^-(x) = -A \operatorname{sech}^2 \alpha x. \quad (3.24)$$

This is another potential for which exact solution of the Schrödinger equation is possible (Landau and Lifshitz 1958). The energy eigenvalues, say  $E_n^-$ , are given by

$$E_n^- = -\gamma^2[(1 + \gamma^{-2}A)^{1/2} - (2n+1)]^2 \quad (3.25)$$

with the same definition of  $\gamma$  as in equation (3.19). The inequality  $E_n > E_n^-$  is satisfied for *all* the bound-state energies even in this case.

#### 4. Counterexamples

We shall now present two counterexamples to show that Steiner symmetrisation of the potential does not, in general, lower the energies corresponding to the excited states. (The ground-state energy, however, is always lowered on Steiner symmetrisation as established in § 2.)

##### 4.1. Two $\delta$ function potentials

Consider the potential  $V(x)$  given by

$$V(x) = -(\hbar^2\alpha/2m)[\delta(x-a) + \delta(x+a)]. \quad (4.1)$$

The solution of the corresponding Schrödinger equation

$$\Psi'' + \alpha[\delta(x-a) + \delta(x+a)]\Psi = K^2\Psi \quad (4.2)$$

where  $K^2 = -2mE/\hbar^2$ , with  $E < 0$ , may be written in the form

$$\Psi(x) = A \exp[-K|(x+a)|] + B \exp[-K|(x-a)|]. \quad (4.3)$$



The above wavefunction is continuous everywhere. However, in the presence of a  $\delta$  function potential the derivative of the wavefunction is not continuous (the probability current density is still continuous as required on physical grounds). The discontinuity in the derivative of the wavefunction is given by

$$\Psi'(b+) - \Psi'(b-) = -\alpha\Psi(b) \quad (4.4)$$

where  $x = b$  is the location of the  $\delta$  function potential. Using the condition (4.4) for  $b = \pm a$ , we get two homogeneous equations determining the constants  $A$  and  $B$ . The condition for a non-trivial solution leads to the following equation for  $K$ :

$$e^{-4Ka} = [(2K/\alpha) - 1]^2. \quad (4.5)$$

It is easy to argue, from considerations of the derivatives at  $K = 0$  of the expressions on both the sides of the above equation, that if  $\alpha a \geq 1$  there are two allowed values of  $K$  and consequently two bound states in the region  $E < 0$ . Also, equation (4.5) has no solution in the range  $\alpha < K < \infty$ , since for such  $K$  the left-hand side is less than unity and the right-hand side is greater than unity.

Upon Steiner symmetrisation of the potential (4.1) we get  $V^-(x)$  as a single  $\delta$  function potential:

$$V^-(x) = -(\hbar^2\alpha/m)\delta(x) \quad (4.6)$$

giving rise to only one bound state corresponding to  $K = \alpha$ , with the energy  $E_0^- = -\hbar^2\alpha^2/2m$ . Thus we have the ground state of  $V^-(x)$  lower than that of  $V(x)$  given by (4.1) with  $\alpha a > 1$ . The first excited state of  $V^-(x)$  is at  $E = 0$ , whereas that corresponding to  $V(x)$  is at some  $E < 0$ . Hence we conclude that the energy level corresponding to the first excited state is not, in general, lowered on Steiner symmetrisation of the potential.

#### 4.2. Asymmetric oscillator

The exact eigenenergies for the potential (3.1) have been tabulated by Ghosh and Hasse (1981) for the two values of  $\alpha \equiv \omega_1/\omega_2 = 0.5, 0.8$ . In table 1 we compare these

**Table 1.** Quantum numbers  $\nu_n^{\text{exact}}$  (Ghosh and Hasse 1981) and  $\nu_n^-$  (corresponding to the Steiner symmetrised potential given by equation (3.7) for two values of  $\alpha$ .

$n$	$\alpha = 0.5$		$\alpha = 0.8$	
	$\nu_n^-$	$\nu_n^{\text{exact}}$	$\nu_n^-$	$\nu_n^{\text{exact}}$
1	0.1667	0.1787	0.0555	0.0567
2	1.5000	1.4934	1.1667	1.1659
3	2.8333	2.8348	2.2778	2.2784
4	4.1667	4.1675	3.3889	3.3886
5	5.5000	5.4990	4.5000	4.5003
6	6.8333	6.8337	5.6111	5.6110
7	8.1667	8.1670	6.7222	6.7224
8	9.5000	9.4997	7.8333	7.8333
9	10.8333	10.8335	8.9444	8.9443
10	12.1667	12.1668	10.0556	10.0556

with the eigenenergies corresponding to the Steiner symmetrised potential (3.7). For ease of comparison we express  $E_n^-$  in terms of  $\nu_n^-$  by writing

$$E_n^- = (\nu_n^- + \frac{1}{2}) \hbar \omega_1 \quad E_n^{\text{exact}} = (\nu_n^{\text{exact}} + \frac{1}{2}) \hbar \omega_1. \quad (4.7)$$

It is clear from table 1 that the inequality  $E_n > E_n^-$  is not valid for all  $n$ . However, the values of  $E_n^-$  are remarkably close to  $E_n$  for both the values of  $\alpha$  and for all the values of  $n$  listed in table 1.

We have also performed a variational calculation for the ground-state energy with potential (3.1) by assuming a trial wavefunction in the form

$$\begin{aligned} \psi &= A \exp(-m\alpha_1 x^2/2\hbar) & x > 0 \\ &= A \exp(-m\alpha_2 x^2/2\hbar) & x < 0 \end{aligned} \quad (4.8)$$

with  $\alpha_1$  and  $\alpha_2$  as variational parameters. Following the standard variational procedure we find

$$E^{\text{var}} = (\hbar/2)(\alpha_1 \alpha_2)^{1/2} \quad (4.9)$$

where

$$\alpha_1 \alpha_2 = \omega_1 \omega_2 [1 + 2(1 - \xi)^2/9\xi]^{1/2} \quad (4.10)$$

with  $\xi \equiv \alpha_2/\alpha_1$ , being the (only) positive root of the biquadratic

$$\xi^4 + 2\xi^3 - 2\alpha^2\xi - \alpha^2 = 0. \quad (4.11)$$

For  $\alpha = 0.5$  we get  $\nu_1^{\text{var}} = 0.1967$  and for  $\alpha = 0.8$  we get  $\nu_1^{\text{var}} = 0.05815$ . These can be compared with the lower bound and the exact value listed in table 1. In table 2 we give the variational upper bound and the lower bound by Steiner symmetrisation for a range of values of  $\alpha$ . It is evident that only values of  $\alpha$  in the interval  $0 < \alpha \leq 1$  need be considered. As expected the agreement is better for the values of  $\alpha$  close to unity.

**Table 2.** Bounds on the quantum number  $\nu_1^{\text{exact}}$  for the asymmetric harmonic oscillator. The variational upper bound  $\nu_1^{\text{var}}$  and the lower bound  $\nu_1^-$ , obtained by Steiner symmetrisation, are given for a set of values of  $\alpha$ .

$\alpha$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$\nu_1^{\text{var}}$	0.8647	0.4650	0.3738	0.2705	0.1967	0.1403	0.0953	0.0581	0.0267	0.0000
$\nu_1^-$	0.4091	0.3333	0.2692	0.2143	0.1667	0.1250	0.0882	0.0556	0.0263	0.0000

We have also performed a numerical calculation of the energy levels where the potential was taken to be a combination of two square well potentials of unequal width and depth. The numerical results clearly show that, on Steiner symmetrisation of the potential, lowering of the energy level occurs for the ground state as well as for a first few excited states but *not for all* the excited states.

## 5. Conclusions

We have shown that the ground-state energy of a Hamiltonian is lowered on Steiner symmetrisation of the potential. We have also shown that, in a number of cases, Steiner

symmetrisation of the potential leads to the lowering of all the bound-state energy levels. In two instances, i.e. a harmonic oscillator on a half-line and a harmonic oscillator with a centripetal barrier, Steiner symmetrisation led to a harmonic oscillator potential. In view of the result derived by Landau and Lifshitz (1960), this should not be surprising. In fact, any classical harmonic potential with a single minimum will be transformed into a harmonic oscillator potential. For this class of potential the ground-state energy of the corresponding Hamiltonians would be bounded from below by the harmonic oscillator ground-state energy.

The Steiner symmetrisation of the Morse potential has brought out its interesting connection with yet another exactly solvable potential.

The idea of Steiner symmetrisation can be applied in a variety of other situations as well. As an example, we may consider the Schrödinger equation corresponding to a two-dimensional potential box which is in the shape of a rhombus. A rhombus can be obtained from a rectangle by Steiner symmetrisation with respect to a diagonal. The rhombus in turn leads to another rectangle under Steiner symmetrisation with respect to a side. Since the Schrödinger equation for a particle in a rectangular box can be solved exactly, we can obtain both an upper and a lower bound on the ground-state energy corresponding to the rhombus-shaped potential box. The same procedure can obviously be extended also to a three-dimensional rhombohedral box.

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